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# ON SOME NONSTANDARD SEMI-INFINITE PROGRAMMING PROBLEMS

BY

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## 1. Introduction

Recently new structures and models of arithmetic developed by A. Robinson [13], [14] and others have provided a logical foundation for a nonarchimedean approach to the Calculus. These structures of nonstandard arithmetic offer "alternatives to familiar infinitary constructions and passages to the limit" (Robinson [14], p. 841). The basis for these methods is embedded in Foundations of Logic and permits consideration of a vast class of number fields which are richer in many ways than the more familiar extended number systems, such as Hilbert's Field of quotients of finite polynomials in one indeterminate.[1] It has long been known, for example, that finite linear programming theory (see [2], [4], and [5]) is valid over any ordered field. Charnes theorem (associating linear independence with extreme points) and the Charnes-Cooper Opposite Sign Theorem ([4]) are valid over any arbitrarily ordered field. Later Charnes-Cooper-Kortanek [8] proved that these two theorems are also valid for generalized finite sequence spaces over any arbitrarily ordered field in the context of semi-infinite programming. Historically, developments of extended number systems in linear programming arose in the context of resolving degeneracy (Charnes [2]) and Goal-programming (see [5], Chapter VI, Appendix B). For recent examples involving economic interpretations see Charnes-Clower-Kortanek [3]. The addition of relative infinitesimals and relative infinites to the real number

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[1] Acknowledgement is due to Professor A. Nerode, Department of Mathematics, Cornell University for bringing these observations to our attention.

field avoids the necessity of considering familiar infinitary constructions in these particular contexts. While much evidence today is being compiled on the importance of extended number fields by outstanding mathematicians, the only textbook discussing linear programming over arbitrarily ordered fields is that of Charnes-Cooper, [5], in particular, pp. 756-757.

The main purpose of this paper is to develop and prove a duality theorem for a special nonstandard semi-infinite programming problem over Hilbert's Field. Attention will center on semi-infinite programs having real coefficients but whose complete regularizations (see [4], [5], and [8]) involve <u>powers</u> of the Nonarchimedean element. A duality result first proved in [12] but never published will be given which states conditions under which the infimum of the primal problem equals the maximum of the dual problem, possibly involving finite powers of the relative infinites. Research is currently under way into the construction of other Nonarchimedean fields in an attempt to obtain strong duality results which resolve, in yet another way, the phenomenon of duality gaps (see [9]).

We first review the basic properties of ordered fields, state the definition of Hilbert's Field, and discuss its topological properties.

## 2. Ordered Fields[1]

A commutative field F is called "ordered" if the property of positiveness (> 0) is defined for all its elements, and if it satisfies the following postulates

1. For every a in F just one of the relations $a = 0$, $a > 0$, $-a > 0$ is valid.

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[1] The results on ordered fields are presented here as in Van der Waerden[15].

2. If $a > 0$ and $b > 0$, then $a + b > 0$ and $ab > 0$.

If the absolute value $|a|$ of an element $a$ in $F$ is defined as the non-negative one of the elements, $a, -a$, then the following usual rules hold:

$$|ab| = |a|\ |b|$$
$$|a + b| \leq |a| + |b|$$

Furthermore, $a^2 = (-a)^2 = |a|^2 \geq 0$ with equality only for $a = 0$. Thus a sum of squares is always positive. For completeness, other properties of inequalities in addition to those above are listed below:[1]

(1) $0 < 1/a \iff a > 0$

(2) $a/b < c/d \iff abd^2 < b^2 cd$

(3) $0 < a < b \implies 0 < 1/b < 1/a$

(4) $a < b < 0 \implies 0 > 1/a > 1/b$

and (5) $a_1^2 + a_2^2 + \ldots + a_n^2 \geq 0$, where all elements here are in $F$.
Again it is emphasized that these rules hold in any ordered field $F$.

Observe that by (5), the characteristic of any ordered field is zero.

If a commutative ring $R$ is ordered, then the quotient field $F$ of $R$ is ordered in a unique way so that the ordering is preserved. Further the ordering of $F$ is given by

$$a/b > 0 \iff ab > 0 \quad {}^2$$

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[1] See Birkhoff and MacLane [1] p. 50.

[2] See Birkhoff and MacLane [1], p. 49 Theorem 12.

The ordering of a field is called Archimedean[1] if there exists a "natural number" $n > a$ for every field element $a$. For example, the ordering of the rational number field is Archimedean. If the ordering of a field is not Archimedean, there exist "infinitely large" elements, larger than any rational number and "infinitely small" elements which are smaller than any positive rational number but larger than zero.

## 3. A Non-Archimedean (Order Preserving) Field Extension of the Real Numbers: The Hilbert Field

Let $R$ be the field of real numbers with the usual ordering and let $U$ be an indeterminate. Define an ordering of the polynomial ring $R[U]$ as follows

$$p \in R\,[U],\ p > 0$$

if and only if the coefficient of the highest nonvanishing power of $U$ is positive.

This defines an ordering in $R\,[U]$ since the laws of trichotomy, addition, and multiplication hold, and the ordering preserves the ordering of $R$. Further the quotient field $R\,[U]$ becomes ordered uniquely from $R\,[U]$ by:

(0) $p/q > 0$ if and only if $pq > 0$,

preserving the ordering of $R\,[U]$ and hence that of $R$.[2]

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[1] See Van der Waerden [15], p. 210.

[2] An equivalent and very intuitive definition of this ordering is given by $p(t) \geq q(t)$ if and only if $q(t)/p(t) < 1$ as $t \to \infty$, where $t$ ranges through real values.

Theorem R (U) under the ordering (0) is non-Archimedean.

Proof For any natural number $n$, we have $n - U < 0$ since the coefficient of the highest power of $U$ is -1. Hence $n < U$; $\therefore$ it is impossible for the Archimedean property to hold.

$U$ is an element of R (U) which is to be regarded as infinitely large, that is, it is larger than any real number. Similarly observe that $1/U$ is infinitely small. Note also that real numbers can be regarded as polynomials in $U$ of $0^{th}$ power. The field R (U) with this ordering is usually called the Hilbert field and it is to this field which a major portion of our attention will be drawn. It is the largest order preserving extension of the real numbers which we consider in this paper.

Again, observe that as far as order relations among elements of R (U) are concerned, they are the same as for the real numbers. Algorithms which use the order operations of real numbers go over into the big field, as is well known for the simplex algorithm.

## 4. The Order Topology of R (U) and the Cauchy Completion

One of the basic properties of the real numbers that is used in semi-infinite programming is completeness, i.e. any Cauchy sequence is convergent. In fact every compact subset of Euclidean n-space is complete. Roughly speaking compactness requires certain limit points to be there and completeness assures that they are in the space itself. Thus we would like the Hilbert field to be complete in a topology that agrees with the subspace of reals.

Now it is known that every ordered field has an ordered extension field which is unique and complete. Of the various constructions of this extension field,perhaps the most well known is Cantor's construction by fundamental sequences.[1] Recall that an infinite sequence $\{a_n\}$ in an ordered field $K$ is <u>fundamental</u> if given $\varepsilon > 0$, and $\varepsilon \in K$, there exists an integer $n$ such that $|a_p - a_q| < \varepsilon$ for $p > n$ and $q > n$. Observe that the topology induced on $K$ by the "$|\ \ |$", which is used in this construction, is the same as the order topology[2], i.e. if the topology for $K$ has a subbase consisting of all sets of the form

$$\{x:x < a\} \text{ or } \{x:a < x\} \text{ for some } a \in K.$$

It follows that the usual topology for the reals is identical with the order topology. Thus in this topology, which is actually a uniform topology, $K$ is uniformly isomorphic to a dense subspace of its completion $K^*$.

<u>Theorem of the Least Upper Bound:</u>[3] If $K$ has an Archimedean ordering, then every non-empty subset of $K^*$ bounded from above has a least upper bound in $K^*$.

In Kelly's terminology $K^*$ is <u>order-complete</u>. As expected if $K = R\ (U)$, then $R^*\ (U)$ is also non-Archimedean and therefore not order complete. Thus there are sets which are closed and bounded but not compact e.g.,

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[1] See Van Der Waerden [15] pp. 212-218.

[2] See Kelly [11] pp. 58-59.

[3] See Van Der Waerden [15] pp. 216-217.

$$A = \{a \in R^* (U) : 0 \leq a \leq U\}.$$

To display the nature of the difficulties that can be encountered in nonstandard programming over $R^* (U)$, consider first the sequence $\{n\}$, $n = 1, 2, \ldots$, in $R^* (U)$. This sequence is not a Cauchy sequence since the minimum distance between members is 1, and therefore no element in $R^* (U)$ is a limit point of this sequence. Consider the system of linear inequalities in one variable over $R^* (U)$,

$$u \geq n \qquad u = 1, 2, \ldots \quad \text{(integers)}$$

Even over $R (U)$, this system is feasible, e.g. $u_o = \alpha U$ with $\alpha > 0$ and real is feasible. The question here however, is whether there can be defined in some sense an extended concept of "canonical closure",[1] so that the duality theory applies as it does if $u$ is required to be real. Two points need emphasis.

(a) Clearly as it stands no inequality over $R^* (U)$ is a limiting inequality of the system since the sequence $\{n\}$, $n = 1, 2, \ldots$ has no limit point in $R^* (U)$.

(b) If we consider the equivalent inequality system (or any one which is bounded in the real sense),

$\frac{1}{n} u \geq 1$, $u = 1, 2, \ldots$, then the limiting inequality $0 \cdot u \geq 1$ should be adjoined for canonical closure. However, with this addition the constraint set over $R^* (U)$ drastically changes, i.e. becomes empty, violating the important property that real systems possess as given previously by the following theorem.

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[1] See [9] for the definition of canonical closure for semi-infinite programming problems.

Theorem. A canonically closed (real) system has the same constraint set as the original system. (See [8] p. 217).

Therefore we conclude that if we wish to maintain the above "constraint set preserving" property of the process of canonical closure, we cannot close up systems over $R^*(U)$ in the same manner as in the real case. In this situation the extended dual theorem, which provides 4 mutually exclusive and collectively exhaustive cases for the dual problems,[1] does not hold as evidenced by our example.

| (I) | (II) |
|---|---|
| $\min u$ | $\max \Sigma n \lambda_n$ |
| $u \geq n, \; n = 1, 2, \ldots$ | $\Sigma \lambda_n = 1$ |
| | $\lambda_n \geq 0 \quad \text{(all } n\text{)}.$ |

(I) is consistent as well as (II), while $\inf\limits_{u \geq n \;\; n = 1, 2 \ldots} u$ does not exist. This exactly displays the lack of order completeness of the extended field $R^*(U)$.

## 5. A Special Nonstandard Semi-Infinite Problem

We consider the following dual semi-infinite programming problems, where $I$ is an arbitrary index set, $\{P_i : i \varepsilon I\} \subset R_m$, $P_0 \varepsilon R_m$, and $c_i \varepsilon R_1$ for all $i \varepsilon I$.[2]

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[1] See [7], [8], and [9].

[2] See [8] and [9] where the Charnes-Cooper device of adding bounding constraints and an extra variable was first applied to arbitrary semi-infinite programs to render them bounded and consistent.

$(I_R)$

$$\min \; u_o M + u^T P_o$$

$$\text{subject to} \quad u_o + u^T P_i \geq c_i \quad \text{all} \quad i \in I$$

$$u^T I_m \geq - U e_m{}^T$$

$$-u^T I_m \geq - U e_m{}^T$$

$$u_o \geq 0$$

$(II_R)$

$$\max \; \Sigma_{i\varepsilon I} \, c_i \lambda_i - U e_m{}^T (\nu^+ + \nu^-)$$

$$\text{subject to} \quad \lambda_o + \Sigma_{i\varepsilon I} \lambda_i = M$$

$$\Sigma_{i\varepsilon I} P_i \lambda_i + I_m (\nu^+ - \nu^-) = P_o$$

$$\text{where} \quad \lambda_o, \lambda_j \geq 0, \; i \in I,$$

$$\text{and} \quad \nu_j{}^+, \nu_j{}^- \geq 0 \; .$$

Since in the theory of semi-infinite programming, the $(\lambda_i)$-vectors have only finitely many non-zero entries, the summation " $\Sigma_{i\varepsilon I}$ " is well defined. Here $I_m$ is the identity matrix of order $m$ and $e_m{}^T$ is the m-vector whose entries are all "ones." Clearly $I_R$ is consistent, for take $u^T = 0$ and $u_o \geq \sup \{c_i\}$, since $\{c_i : i \in I\}$ is a compact set. As for $(II_R)$, take $\lambda_i = 0 \; (i \in I)$ and $\nu^+ - \nu^- = P_o$ with $\lambda_o = M$. We shall assume that the set of coefficients $\{(P_i, c_i) : i\varepsilon I\}$ is compact in $R_{m+1}$ and that if the system $u^T P_i \geq c_i$, $i\varepsilon I$, is consistent, then it has an interior point. Then by the Extended Dual

Theorem ([8] and [9]) it follows that there exist vectors $\lambda^*$, $\nu^*$ such that $\inf \{u_o M + u^T P_o\} = \Sigma_{i\varepsilon I}\, c_i\lambda_i{}^* - U\, \Sigma_{j=1}^{m}\, \nu_j{}^*$, where $\nu_j = \nu_j{}^+ - \nu_j$. Further, the relevant set[1] $(u_o, u^T)$ for $(I_R)$ is non-empty and compact since it is the intersection of closed sets (half-spaces) and by regularization is non-empty and bounded. Thus by compactness the "inf" is actually assumed for some $u_o{}^*$, $u^{*T}$.

We propose to study problems $(I_R)$ and $(II_R)$ by relaxing $U$ and $M$ through real values and observing the behavior of the solutions. Thus, we shall take a sequence of positive numbers, $U_n \to \infty$ as $n \to \infty$ and set $M_n = U_n{}^2$. We claim that in the limit it is possible to attain a non-Archimedean dual equation, where $U \varepsilon R\ (U)$, of a special type gotten from regularizing the minimization problem in powers of $U$. The non-Archimedean dual equation really represents a summary equation of the types (or orders) of divergence to infinity that may exist between unbounded variables. In order to provide for this variation, it is necessary to introduce powers of the indeterminate $U$.

Before we state and prove the theorem we turn to some notions of order as developed by Hardy [1].

Assume that $f$ and $g$ are positive and continuous. We say the order of $f$ is greater than that of $g$ if $f/g \to \infty$ when $x \to \infty$ and we write $f \succ g$.[2] If $f/g \to 0$, we say that the order of $f$ is less

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[1] Only bounded $u_o$ are relevant.

[2] See Hardy [10] P. 1-5.

than that of $g$ and write $f \prec g$. If $f/g$ remains bounded for all values of $x$ after a certain point, then we say that the order of $f$ equals the order of $g$ and write $f \succ\!\prec g$. If $f/g$ tends to a definite limit, we write $f \asymp g$. If $f$ and $g$ are <u>monotonic</u> as well as $f/g$, then the following trichotomy holds:

$$\text{(TR)} \quad \text{either} \quad f \succ g \quad \text{or} \quad f \prec g \quad \text{or} \quad f \asymp g.$$

Similarly we may introduce order notions for negative functions, which tend to $-\infty$, and we use the same symbols, e.g. $-\log x \prec -x$ having the same meaning as $\log x \prec x$.

With this introduction we now state our theorem.

<u>Theorem</u> Assume that the semi-infinite system $(I_R)$ has a compact set of coefficients in $R_{m+1}$. Then there exist non-Archimedean dual optimal solutions to $(I_R)$ and $(II_R)$ respectively involving only powers of the indeterminate $U$ and $P_i \in R_m$, such that

$$u_o^* M + u_*^T P_o = \sum_{i \in I} c_i \lambda_i^* - \sum_{j=1}^{m} U^{\alpha(j)} \nu_j^*$$

where $\alpha : [1, 2, \ldots, m] \to [0, 1, \ldots m]$, $M = U^{m+1}$, and the base field is $R\,(U)$.

We may speak of the semi-infinite program min $u^T P_o$, subject to $u^T P_i \geq c_i$ all $i \epsilon I$ as being <u>embedded</u> in a non-Archimedean extension. Each real semi-infinite program may in general have a different non-Archimedean embedding. From the viewpoint of the process of forming regularizations, the theorem states that it is possible to do non-Archimedean semi-infinite programming of a very special type, namely,

a "regularized" problem that originates from a compact semi-infinite problem with vectors having only <u>real</u> entries. The idea of the proof is to attain a finite non-Archimedean problem having only a finite number of half-spaces as constraints which are either limits of the original ones, $(P_i, c_i)$ or involve powers of the indeterminate $U$ as a bound. The limiting case will then be a finite non-Archimedean problem to which we can apply known <u>finite</u> duality theory.

<u>Proof</u>: We may simplify notation at the outset by omitting the variable $u_o$ and assume the problem to be <u>consistent</u> with interior point over the reals. This is because only bounded $u_o$ are relevant, and $u_o$ is introduced only to provide consistency. Let $Q(U)$ denote the cube $u^T I_m \geq - U e^T_m$, $-u^T I_m \geq - U e^T_m$, with the understanding that $U$ always denotes the indeterminate. Let $U_n$ be a real positive sequence $U_n \to \infty$ as $n \to \infty$.

Consider the following sequence of minimization problems as $U_n \to \infty$:

$$\min u^T P_o$$

$$\text{subject to} \quad u^T P_i \geq c_i \ (i \in I)$$

$$u \in Q\ (U_n)$$

Optimal solutions exist by compactness or Haar duality. Let $u^{(n)}$ be an optimal solution and let $T_n$ be a maximal linearly independent set of tangent planes from the collection $\{(P_i, c_i): i \in I\}$, i.e., $P_i \in T_n$ means ${u^{(n)}}^T P_i = c_i$. Furthermore, at each stage, the associated finite problem,

$$\min u^T P_o$$

$$\text{subject to} \qquad u^T P_i \geq c_i \qquad \text{all} \quad P_i \in T_n$$

$$u \in Q\ (U_n)$$

has $u^{(n)}$ as an optimal solution. Since the collection of P-planes is compact, there exists a limiting set $\{P_{\infty(1)}, \ldots, P_{\infty(k)}\}$ of (maximal) linearly independent vectors of the sets $T_n$ as $n \to \infty$.

Now as $n \to \infty$, $u^{(n)}$ will exhibit coordinates which are bounded and coordinates which are not. Let $S \equiv$ coordinate positions which are bounded, $W \equiv$ coordinate positions which are not bounded. The bounded coordinates will have limit points, say $u_i^{(\infty)}$ for all $i \in S$. Thus, at the $n^{th}$ stage of the procedure (regularization with $u \in Q\ (U_n)$), the optimal solution will have the form,

$$u^{(n)} = \begin{cases} u_j^{(n)} : j \in S \\ f_j^{(n)} : j \in W \end{cases}$$

where $u_j^{(n)}$ and $f_j^{(n)}$ are, of course, real numbers. However, the $u_j^{(n)}$'s approach limits, while the $f_j^{(n)}$'s do not. Observe that feasibility is expressed as follows:

$$\sum_{j \in S} u_j^{(n)} a_j^{(i)} + \sum_{j \in W} f_j^{(n)} a_j^{(i)} \geq c_i, \ i \in I \quad \text{where} \quad P_j = (a_j^{(i)}).$$

We need some lemmas about the respective orders of the $f_j$'s.

<u>Lemma 2</u>: The trichotomy (TR) introduced above holds for the $f_j$'s. That is, for $i \neq j$ in $W$ one of the following hold:

$$f_i \succ f_j \quad \text{or} \quad f_i \prec f_j \quad \text{or} \quad f_i \asymp f_j.$$

Proof: At some large $n^{th}$ stage, the approximate order of greatest decrease for minimization will be established, i.e., if the dominance of $f_i^{(n)}$ over $f_j^{(n)}$ is observed, it cannot reverse itself at some later stage since the $T_n$-planes are approaching a limit set, and the finite coordinates (if there are any) of the optimal $u^{(n)}$ are also approaching limits. If at some large stage n, $f_i^{(n)} > f_j^{(n)}$, then at further stages the relative direction of $f_i^{(n)}$ will be sought, only possibly to be blocked by some finite plane in the $T_n$-set. However, at further stages these planes approach limits and their changes and effects on such limitation are negligible. This also shows the continuity of the functions $f_j$.

Thus, we may partition the index set $W$ into subsets $W^{(h)} \subset W$ such that each subset corresponds to functions of the same order, i.e.

$$W^{(1)} \cup W^{(2)} \cup \; \ldots \; \cup W^{(k)} = W$$

and functions in $W^{(i)}$ are of smaller order than those in $W^{(j)}$ if $i < j$.

For each order class choose a (positive) representative $\phi^{(h)}$, $h \in W^{(h)}$ in terms of which other members of the class differ by multiplicative constants in the large, say $v_j$. Thus, in terms of the partitioning of $W$, feasibility may be expressed as follows at the $n^{th}$ stage:

$$(*) \quad \sum_{j \in S} u_j^{(n)} a_j^{(i)} + \sum_{j \in W(1)} (1 + v_j)\, \phi^{(1)} a_j^{(i)} + \ldots$$

$$+ \sum_{j \in W(k)} (1 + v_j)\, \phi^{(k)} a_j^{(i)} \geq c_i \quad \text{for } i \in I.$$

These inequalities may only be approximate, in which case the error $\to 0$ as $n \to \infty$. This is not to say feasibility is attained with real values.

<u>Lemma 3</u>: In the above partitioning of $W$ as expressed in the (*) inequalities, we may replace the $\phi^{(h)}$ representatives by $\pm U^h$, where $U \in R(U)$, and attain feasibility.

<u>Proof</u>: Observe that in each class the coefficients $\sum_{j \in W^{(h)}} (1 + v_j)$ are essentially constant and express the order relations between functions within the class. Two functions of the same order may differ by a multiplicative constant "at ∞." We now proceed by induction. The induction assumption is as follows. For $\leq k - 1$ order classes in $W$ it is possible to assign appropriate powers to attain feasibility. For the case $k = 1$, feasibility is expressed at the $n^{th}$ stage by

$$\sum_{j \in S} u_j^{(n)} a_j^{(i)} + f_n a_w^{(i)} \geq c_i,$$

where $f_n$ is unbounded. Since $\{c_i\}$ is compact as well as $\{a_j^{(i)} : j \in S\}$ and $u_j^{(n)} \to u_j^{(\infty)}$, it follows that the choice of $U$ or $-U$ (depending on which direction $f_n$ moves) maintains the system of inequalities, even with the limit variables $u_j^{(\infty)}$ replacing $u_j^{(n)}$, $j \in S$. For the case $k = 2$, we are confronted with a situation as follows:

$$f_1 a_1^{(i)} + f_2 a_2^{(i)} \geq c_i$$

all $i \in I$, where $f_1 \succ f_2$ and bounded variables have been omitted. We must show, that $U^2 a_1^{(i)} + U a_2^{(i)} \geq c_i$, $i \in I$.[1] Suppose not, i.e., $U^2 a_1^{(i_o)} + U a_2^{(i_o)} < c_{i_o}$, for some $i_o$. First, $a_1^{(i_o)} = a_2^{(i_o)}$ leads to a contradiction, $0 \geq c_{i_o}$ and $0 < c_{i_o}$. Therefore either $a_1^{(i_o)} < 0$

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[1] Here we have assumed for simplicity that the coefficients have absorbed possible negative signs so that only directions at (+ ∞) are relevant.

or $a_1^{(i_o)} = 0$ and $a_2^{(i_o)} < 0$, which again is impossible since $f_1 \succ f_2$ and coefficients of the system are compact. Thus, $U^2 a_1^{(i)} + U\, a_2^{(i)} \geq c_i$, $i \in I$. Now assume that for $\leq k - 1$ unbounded variables the required assignment is possible, and consider the case for $k$ order classes. Again, at the $n^{th}$ stage, feasibility is expressed by,

$$\sum_{j \in S} u_j^{(n)} a_j^{(i)} + \sum_{j \in W^{(1)}} f_j^{(1)}\, a_j^{(i)} + \ldots + \sum_{j \in W^{(k)}} f_j^{(k)} a_j^{(i)} \geq c_i, \; i \in I.$$

In terms of the representative $\phi^{(k)}$ of $W^{(k)}$ we cannot have $\sum_{j \in W^{(k)}} (1 + v_j)\, a_j^{(i)} < 0$ in the limit because this class is dominant and would destroy feasibility in such a situation. Thus, the net effect, $\sum_{j \in W^{(k)}} (1 + v_j)\, a_j^{(i)}$ is positive, and if strictly positive, the assignment of $U^k$ (or $-U^k$) will maintain feasibility. Further, if the net effect is zero, the induction assumption now applies to the remaining $k - 1$ variables. Therefore the indicated substitutions of $U$ powers as expressed in the lemma are valid.

Thus, for all $n$ beyond some fixed point, feasibility is attained with the U-power substitutions. In the limit we get an optimal solution with coordinates $u_i^{(\infty)}$, $i \in S$ and $U$ - power coordinates for $i \in W$, which is determined by the $T_\infty$ set and the U - power bounding hyperplanes. Taken together these hyperplanes form a finite non-Archimedean problem with the same minimum as the semi-infinite one due to our construction above. Hence, <u>finite</u> non-Archimedean duality theory now applies to

provide dual feasibility and the dual equation of the form required of the theorem. Observe that inconsistency is handled by bringing in the additional variable $u_o$ with functional value larger than any of the powers of $U$ assigned to unbounded variables, e.g. with coefficients $U^{m+1}$. This completes the proof of the Theorem.

We conclude with three examples of problems whose variables have different rates of growth to $\pm\infty$.

<u>Example 1</u>

$$\min\ u_o + u_1$$

$$\text{subject to}\quad \begin{aligned} u_o &\geq 0 \\ u_o + 1/k\ u_1 + 1/k^2\ u_2 &\geq 0, \quad \text{for } k = 2, 3, \ldots\ . \end{aligned}$$

This system has interior points and the coefficient set is compact. Therefore it is canonically closed. Further, if we regularize with $U_n$ (real) large, we find the solution to be

$$u_o^* = 1/2^2\ U_n,\ u_1^* = -U_n,\ \text{ and }\ u_2^* = U_n.$$

These all have the same order, and therefore there is only one order class for unbounded coordinate positions. In fact, adjoining linear powers of $U$ as constraints yields the optimum $u_o^* = 1/2^2\ U$, $u_1^* = -U$, and $u_2^* = U$. Furthermore, the regularized dual reads:

$$\max\quad - U\,[\nu_1^+ + \nu_2^+ + \nu_1^- + \nu_2^-]$$

$$\text{subject to}\quad \begin{pmatrix}1\\0\\0\end{pmatrix}\lambda_o + \Sigma\begin{pmatrix}1\\1/k\\1/k^2\end{pmatrix}\lambda_k + \begin{pmatrix}0\\1\\0\end{pmatrix}\nu_1^+ + \begin{pmatrix}0\\0\\1\end{pmatrix}\nu_2^+ + \begin{pmatrix}0\\-1\\0\end{pmatrix}\nu_1^- + \begin{pmatrix}0\\0\\-1\end{pmatrix}\nu_2^- = \begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$\text{and }\ \lambda\text{'s},\ \nu\text{'s} \geq 0.$$

The dual optimum reads: $\lambda_2^* = 1$, $\nu_1^{+*} = 1/2$, $\nu_2^{-*} = 1/2^2$ with dual functional, $-U\ (1/2 + 1/2^2) = -\ 3/4\ U$. Since the direct functional is $u_o^* + u_1^* = -\ 3/4\ U$, we attain the dual equation.

<u>Example 2.</u>

$$
\begin{array}{lrcl}
\min & & -\ u_2 & \\
 & u_1\ 1/x & -\ u_2 & \geq 1 - \log x,\ x \geq 2 \\
 & u_1 & & \geq 1 \\
 & 0\ u_1 & -\ 0\ u_2 & \geq -1 \\
 & -\ u_1 & & \geq -\ U^2 \\
 & & -\ u_2 & \geq -\ U
\end{array}
$$

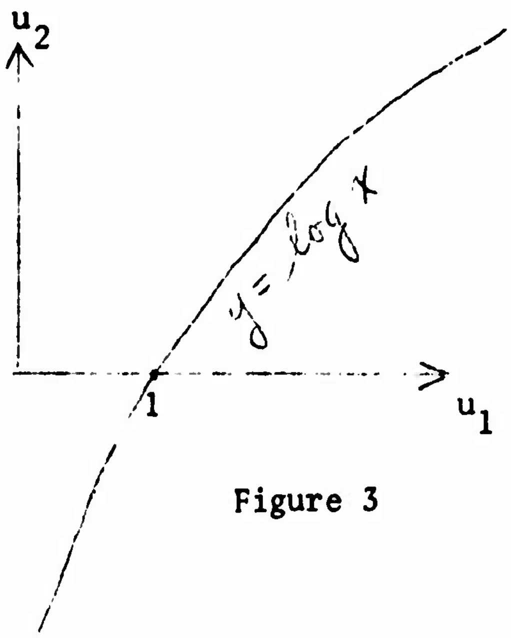


Figure 3

Regularizing with $U_n$ yields an optimum

$$u_1^* = U_n \quad \text{and} \quad u_2^* = \log U_n.$$

In this case there are two order classes, and variable $u_2$ is regularized with respect to $U$ and $u_1$ with respect to $U^2$. Thus, the non-Archimedean optimum reads $u_1^* = U^2$, $u_2^* = U$. The dual problem has an optimum at $\nu_2^{-*} = 1$ and all other variables zero, with equation of dual functionals prevailing, i.e.

$$\max \sum_x (1 - \log x)\ \lambda_x + \lambda_1 - \lambda_2 - U^2\ \nu_1^- - U\ \nu_2^-$$

$$\text{s.t.} \quad \sum_x \begin{pmatrix} 1/x \\ -1 \end{pmatrix} \lambda_x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \lambda_1 + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \nu_1^- + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \nu_2^- = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\lambda_x,\ \lambda_1,\ \lambda_2,\ \nu_1^-,\ \nu_2^- \geq 0.$$

Example 3[1] Consider as the constraint set the points under the curve $y = \tan^{-1}(x)$, with $x \geq 0$ and above the x-axis.

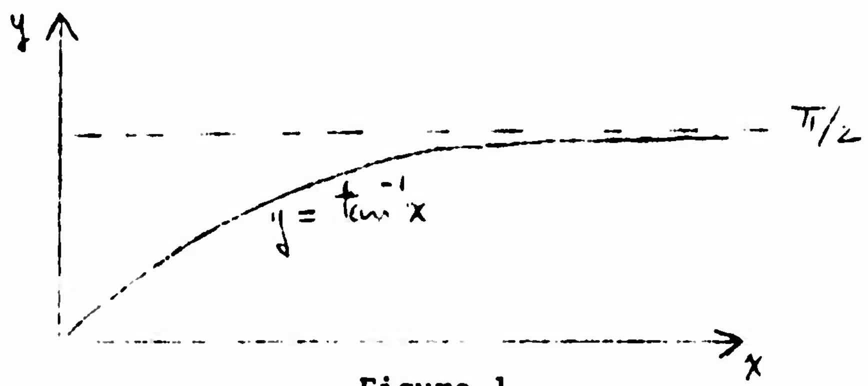


Figure 1

We observe that the equation of the tangent line at the point $(x, \tan^{-1} x)$ is given by $(u_2 - \tan^{-1} x)/(u_1 - x) = 1/(1 + x^2)$ and therefore our constraint set is given by the following system of inequalities

(I)

$$\begin{aligned} u_1 (1 + x^2)^{-1} - u_2 &\geq - \tan^{-1} x + x (1 + x^2)^{-1} = c_x \\ u_1 \qquad\qquad &\geq 0 \\ - u_2 &\geq - (\pi/2) \end{aligned}$$

Let the direct problem, I, be min $(-u_2)$ subject to the above constraints. In this example the set of coefficient points of

$$\{(1 + x^2)^{-1}, -1, -\tan^{-1} x + x (1 + x^2)^{-1}) : x \geq 0\}$$

in addition to the points $(1, 0, 0)$ and $(0, -1, -\pi/2)$. Clearly this set is compact, since the limit point $(0, -1, -\pi/2)$ as $x \to \infty$ is added. Note that min $-u_2 = -\max u_2 = -\pi/2$ is never attained because feasible points must lie on or under the curve.

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[1] See [8] pp. 214-215.

For this example, the dual II is as follows.

If we let

$$P_x = \begin{pmatrix} (1 + x^2)^{-1} \\ 1 \end{pmatrix}, \; P_\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; P_\beta = \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

then we have

$$\max \Sigma\, c_x \lambda_x + 0 \cdot \lambda_\alpha + (-\pi/2)\, \lambda_\beta$$

subject to

$$\Sigma\, P_x\, \lambda_x + P_\alpha \lambda_\alpha + P_\beta \lambda_\beta = P_o = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad \text{all } \lambda\text{'s} \geq 0.$$

We see that if $\lambda_\beta = 1$ and all other $\lambda$'s $= 0$, then the maximum $-\pi/2$ is attained. Observe, however, that if we adjoin the constraint $u_2 \geq \pi/2$, then problem (I) becomes <u>inconsistent</u> over the real field. The dual solution, however, prevails. Over the Hilbert field R (U), as previously introduced, problem (I) is consistent, e.g., $u_1 = U$ and $u_2 = \pi/2$. We view this type of inconsistency or duality gap over the real field as one which may be removed by considering the semi-infinite problem in a nonstandard way, namely programming over R (U).

## References


[1] Birkhoff, G., and S. MacLane, A Survey of Modern Algebra, The Macmillan Company, New York, 1953.

[2] Charnes, A., "Optimality and Degeneracy in Linear Programming," Econometrica, Vol. 20, No. 2, pp. 160-170.

[3] Charnes, A., R. W. Clower, and K. Kortanek, "Effective Control Through Coherent Decentralization with Preemptive Goals," forthcoming Econometrica.

[4] Charnes, A. and W. W. Cooper, "The Strong Minkowski-Farkas-Weyl Theorem for Vector Spaces over Ordered Fields," Proc. Nat. Acad. Sci. U.S.A., 44, No. 9, Sept. 1958, pp. 914-916.

[5] Charnes, A. and W. W. Cooper, Management Models and Industrial Applications of Linear Programming, Vols. I and II, New York, J. Wiley and Sons, 1961.

[6] Charnes, A., W. W. Cooper and K. Kortanek, "Duality, Haar Programs, and Finite Sequence Spaces," Proc. Nat. Acad. Sci. USA, 48, No. 5, May 1962, pp. 783-786.

[7] Charnes, A., W. W. Cooper and K. Kortanek, "A Duality Theory for Convex Programs with Convex Constraints," Bull. of the Amer. Math. Soc., Vol. 68, No. 6, Nov. 1962, pp. 605-608.

[8] Charnes, A., W. W. Cooper and K. Kortanek, "Duality in Semi-infinite Programs and Some Works of Haar and Caratheodory," Management Science, 9, No. 2, Jan. 1963, pp. 209-228.

[9] Charnes, A., W. W. Cooper and K. Kortanek, "On Representations of Semi-Infinite Programs Which Have No Duality Gaps," Management Science, Vol. 12, No. 1, Sept. 1965.

[10] Hardy, G. H., Orders of Infinity, Cambridge Tracts in Mathematics and Mathematical Physics, No. 12, Cambridge University Press, 1954.

[11] Kelley, J. L., General Topology, D. Van Nostrand Co., Inc. 1955.

[12] Kortanek, K. O., "Duality, Semi-Infinite Programming, and Some Aspects of Control in Business and Economic Systems, Ph.D. Thesis, Northwestern University, Department of Engineering Sciences, Evanston, June, 1964.

[13] Robinson, A., Non-Standard Analysis, Studies in Logic and the Foundations of Mathematics, North-Holland, Amsterdam, 1966.

[14] Robinson, A., "Non-standard Arithmetic",Bulletin of the American Mathematical Society, Vol. 73, No. 6, Nov. 1967.

[15] Van der Waerden, B. L., Modern Algebra, Vol. I, Frederick Ungar Publishing Co., New York, 1953.

[16] Springer, T.A. (ed.) Proceedings of a Conference on Local Fields
a. Monna, A.F. Linear Topological Spaces over Non-Archimedean Valued Fields

b. Remmert, R. Algebraische Aspekte in der nichtarchimedischen Analysis.

Springer, New York, 1967.

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13. ABSTRACT

Aspects of the duality theory for semi-infinite programming are here extended to fields with properties of non-Archimedean order. Emphasis is on nonstandard semi-infinite programming problems in Hilbert's Field. The ideas of regularization are generalized to include powers of the relative infinites in terms of the indeterminates.

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